General Solution For Non-Homogeneous Ordinary Differential Equation With Constant Coefficient

The General Form Of Second Order Ordinary Differential Equation Which is Non-Homogeneous

is as Following:

$$y'' + ay' + by = f(x) \leftarrow \text{NON-Hom}$$

- the solution of equation depend on $f(x)$

- Firstly start with $\boxed{f(x) = e^{\alpha x}}$

The general solution of equation can be represented as:

$$\boxed{y = y_h + y_p}$$

$y_{homogeneous}$

→ To get y_h → we ignore $f(x)$

so, we solve:

$$\boxed{y'' + ay' + by = 0}$$

→ To get y_p, we rewrite equation as following:

$$\boxed{y'' + ay' + by = e^{\alpha x}}$$

$\because D = \frac{d}{dx} \rightarrow D^2 = \frac{d^2}{dx^2}$

$$D^2 y + aDy + by = e^{\alpha x}$$

$$[D^2 + aD + b]\, y = e^{\alpha x}$$

$$y = \frac{1}{D^2 + aD + b} e^{\alpha x}$$

Then we replace D with α

$$\boxed{y_p = \frac{1}{\alpha^2 + a\alpha + b} e^{\alpha x}}$$

then final solution:

$$\boxed{y = y_h + y_p}$$

Ex: Solve following ordinary differential equation:

1) $y'' + y' - 6 = 8e^{3x}$

Sol:

$$y = y_h + y_p$$

To get y_h → (ignore right hand side)

$$y'' + y' - 6 = 0 \leftarrow \text{(Homogeneous)}$$

- we assume that the solution can be represented as

$y = e^{Dx}$ (D is unknow must be Determined)

→ $y' = De^{Dx}$ → $y'' = D^2 e^{Dx}$

$$D^2 e^{Dx} + De^{Dx} - 6e^{Dx} = 0$$

$$e^{Dx}[D^2 + D - 6] = 0$$

$$D^2 + D - 6 = 0$$

$$(D+3)(D-2) = 0$$

$D = -3$ $D = 2$

$$y_h = C_1 e^{2x} + C_2 e^{-3x}$$

Ex: Solve following ordinary differential equation:

1) $y'' + y' - 6 = 8e^{3x}$

Sol:

$$y = y_h + y_p$$

To get y_h → (ignore right hand side)

$$y'' + y' - 6 = 0 \leftarrow \text{(Homogeneous)}$$

- we assume that the solution can be represented as

$$y = e^{Dx} \quad (D \text{ is Unknow must be Determined})$$

$$\rightarrow y' = De^{Dx} \quad \rightarrow y'' = D^2 e^{Dx}$$

$$D^2 e^{Dx} + De^{Dx} - 6e^{Dx} = 0$$

$$e^{Dx}[D^2 + D - 6] = 0$$

$$D^2 + D - 6 = 0$$

$$(D+3)(D-2) = 0$$

$$D = -3 \qquad D = 2$$

$$y_h = C_1 e^{2x} + C_2 e^{-3x}$$

- To get y_p

$$y'' + y' - 6y = 8e^{3x}$$

$$\leftarrow D = \frac{d}{dx}$$

$$D^2y + Dy - 6y = 8e^{3x}$$

$$D^2 = \frac{d^2}{dx^2}$$

$$[D^2 + D - 6]\, y = 8e^{3x}$$

$$y = \frac{1}{[D^2+D-6]}(8e^{3x})$$

replace $D \to 3$

$$y = \frac{1}{(3^2+3-6)}(8e^{3x})$$

$$y_p = \frac{4}{3}e^{3x}$$

$$\Rightarrow y = y_h + y_p$$

$$y = c_1e^{2x} + c_2e^{-3x} + \frac{4}{3}e^{3x}$$

$\boxed{2}$ $y'' - 9y = 2e^{4x}$

$y = y_h + y_p$

→ To get y_h (we ignore right hand side)

we assume $y = e^{Dx} \rightarrow y' = De^{Dx} \rightarrow y'' = D^2 e^{Dx}$

$D^2e^{Dx} - 9e^{Dx} = 0$

$e^{Dx}[D^2 - 9] = 0$

$D^2 - 9 = 0$

$(D+3)(D-3) = 0$

$D_1 = -3 \; , \; D_2 = 3$

$y_h = c_1 e^{-3x} + c_2 e^{3x}$

→ To get y_p

$y'' - 9y = 2e^{4x}$

$D = \frac{d}{dx}$

$D^2 y - 9y = 2e^{4x}$

$D^2 = \frac{d^2}{dx^2}$

$[D^2 - 9]\, y = 2e^{4x}$

We replace $D \to 4$

$$y_p = \frac{1}{4^2 - 9} \cdot 2e^{4X}$$

$$y_p = \frac{2}{7} e^{4X}$$

$$\therefore y = y_h + y_p$$

$$y = c_1 e^{3X} + c_2 e^{-3X} + \frac{2}{7} e^{4X}$$

$\boxed{3}$ $y'' + 3y' + 2y = e^{5X}$

$\rightarrow$ To get y_h (ignore R.H.S)

$$\therefore y = e^{DX} \rightarrow y' = De^{DX} \rightarrow y'' = D^2 e^{DX}$$

$$D^2 e^{DX} + 3De^{DX} + 2e^{DX} = 0$$

$$e^{DX}\,[D^2+3D+2]=0$$

$$D^2+3D+2=0$$

$$(D+1)(D+2)$$

$$D_1=-1 \quad , \quad D_2=-2$$

$$y_h = c_1 e^{-X} + c_2 e^{-2X}$$

$\rightarrow$ To get $y_p \Rightarrow$

$D = \frac{d}{dx}$

$D^2 = \frac{d2}{dx}$

$$D^2y + 3Dy + 2y = e^{5X}$$

$$[D^2+3D+2]\,y = e^{5X}$$

$$y = \frac{1}{(D^2+3D+2)} \cdot e^{5X}$$

we replace $D \rightarrow 5$

$$y = \frac{1}{42} e^{5X}$$

$y = y_h + y_p$

$$y = c_1 e^{-X} + c_2 e^{-2X} + \frac{1}{42} e^{5X}$$

$\boxed{4}$ $y'' - y' - 2y = e^{x}$

$\therefore$ To get $y_h \rightarrow$ (ignore R.H.S)

$y = e^{Dx} \rightarrow y' = De^{Dx} \rightarrow y'' = D^2 e^{Dx}$

$D^2 e^{Dx} - De^{Dx} - 2e^{Dx} = 0$

$e^{Dx}[D^2 - D - 2] = 0$

$(D - 2)(D+1) = 0$

$D_1 = 2, \quad D_2 = -1$

$\therefore y_h = c_1 e^{2x} + c_2 e^{-x}$

$\rightarrow$ To get $y_p \rightarrow$

$\therefore$ $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$

$D^2 y - Dy - 2y = e^{x}$

$[D^2 - D - 2]\, y = e^{x}$

$y = \frac{1}{(D^2 - D - 2)} e^{x}$

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we replace $D \to 1$

$$y = \frac{-1}{2} e^{x}$$

$$\therefore y = y_h + y_p$$

$$y = c_1 e^{-x} + c_2 e^{2x} - \frac{1}{2} e^{x}$$

$\boxed{5}$ $y''' - 5y'' + 5y' - 25y = e^{2x}$

$\to$ To get y_h $\to$ (ignore R.H.S)

$\because y = e^{Dx} \to y' = D e^{Dx} \to y'' = D^2 e^{Dx} \to y''' = D^3 e^{Dx}$

$$D^3 e^{Dx} - 5D^2 e^{Dx} + 5D e^{Dx} - 25 e^{Dx} = 0$$

$$e^{Dx}[D^3 - 5D^2 + 5D - 25] = 0$$

$$D^3 - 5D^2 + 5D - 25 = 0$$

$$D^2(D-5) + 5(D-5) = 0$$

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$(D-5)(D^2+5)$

$D=5 \mid D^2=-5$

$D_1=5 \qquad D_{2,3}=\pm\sqrt{5}\,i$

$\alpha=0\,, \quad \beta=\sqrt{5}$

$y_h = c_1 e^{5x} + e^{0x}[c_2 \cos\sqrt{5}x + c_3 \sin\sqrt{5}x]$

$y_h = c_1 e^{5x} + c_2 \cos\sqrt{5}x + c_3 \sin\sqrt{5}x$

→ To get y_p .

$D^3y - 5D^2y + 5Dy - 25y = e^{2x}$

$[D^3-5D^2+5D-25]y = e^{2x}$

$$y = \frac{1}{D^3-5D^2+5D-25} \cdot e^{2x}$$

replace $D \to 2$

$$y_p = \frac{1}{8-20+10-25} \cdot e^{2x}$$

$$y_p = \frac{-1}{27} e^{2x}$$

$$\therefore \boxed{y = c_1 e^{5x} + c_2 \cos\sqrt{5}x + c_3 \sin\sqrt{5}x + \frac{-1}{27} e^{2x}}$$

(6) $y''' - 5y' - 2y = e^{3x}$

- To get y_h → we ignore R.H.S

$$y = e^{Dx} \rightarrow y' = De^{Dx} \rightarrow y'' = D^2 e^{Dx} \rightarrow y''' = D^3 e^{Dx}$$

$$D^3 e^{Dx} - 5De^{Dx} - 2e^{Dx} = 0$$

$$e^{Dx}[D^3 - 5D - 2] = 0$$

$$D^3 - 5D - 2 = 0$$

$$(D+2)(D^2 - 2D - 1) = 0$$

$D_1 = -2$

$$D_{2,3} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$= \frac{2 \pm \sqrt{4 + 4}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

$$D_{2,3} = 1 \pm \sqrt{2}$$

$$y_h = c_1 e^{-2x} + c_2 e^{(1+\sqrt{2})x} + c_3 e^{(1-\sqrt{2})x}$$

→ To get y_p ⇒

$$D^3 y - 5Dy - 2y = e^{3x}$$
$$[D^3 - 5D - 2]\, y = e^{3x}$$

$$y = \frac{1}{(D^3 - 5D - 2)} \cdot e^{3x}$$

we replace $D \to 3$

$$y = \frac{1}{3^3 - 5\times 3 - 2} \cdot e^{3x}$$

$$y_p = \frac{1}{10} \cdot e^{3x}$$

$$\therefore y = y_h + y_p$$

$$y = c_1 e^{-2x} + c_2 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x} + \frac{1}{10} e^{3x}$$

$\boxed{7}$ $y''' - 4y' = e^{x}$

- To get y_h → (we ignore R.H.S)

$y = e^{Dx} \rightarrow y' = De^{Dx} \rightarrow y'' = D^2 e^{Dx} \rightarrow y''' = D^3 e^{Dx}$

$$D^3 e^{Dx} - 4De^{Dx} = 0$$

$$e^{Dx}(D^3 - 4D) = 0$$

$D^3 - 4D = 0$

$D(D^2 - 4) = 0$

$D(D-2)(D+2) = 0$

$D_1 = 0 , D_2 = 2 , D_3 = -2$

$\Rightarrow y_h = c_1 + c_2 e^{2x} + c_3 e^{-2x}$

$\rightarrow$ To get y_p

$D^3 y - 4Dy = e^x$

$[D^3 - 4D]\, y = e^x$

$$y_p = \frac{1}{D^3 - 4D} e^x$$

replace $D \rightarrow 1$

$y_p = -e^x$

$$\boxed{y = c_1 + c_2 e^{2x} + c_3 e^{-2x} - e^x}$$

[8] $y''' + 2y'' + 4y' = e^{2x}$

$\rightarrow$ To get $y_h \rightarrow$ (we ignore R.H.S)

$y = e^{Dx} \rightarrow y' = De^{Dx} \rightarrow y'' = D^2 e^{Dx} \rightarrow y''' = D^3 e^{Dx}$

$D^3 e^{Dx} + 2D^2 e^{Dx} + 4De^{Dx} = 0$

$e^{Dx}[D^3 + 2D^2 + 4D] = 0$

$D[D^2 + 2D + 4] = 0$

$D_1 = 0$

$$D_{2,3} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= -1 \pm \sqrt{3}\, i$$

$\alpha = -1$, $\beta = \sqrt{3}$

$y_h = c_1 + e^{-x}[c_2 \cos\sqrt{3}\, x + c_3 \sin\sqrt{3}\, x]$

→ To get y_p

$$D^3y + 2D^2y + 4Dy = e^{2x}$$

$$[D^3 + 2D^2 + 4D]\, y = e^{2x}$$

$$y_p = \frac{1}{D^3 + 2D^2 + 4D} e^{2x}$$

we replace $D \to 2$

$$y_p = \frac{1}{24} e^{2x}$$

$$\therefore y = y_h + y_p$$

$$\boxed{y = C_1 + e^{-x}[C_2 \cos\sqrt{3}x + C_3 \sin\sqrt{3}x] + \frac{1}{24}e^{2x}}$$