The General Solution of Homogeneous Linear Differential Equation With Constant Coefficient

* Second order ordinary differential equation with constant Coefficients:

$$y'' + a y' + b y = 0 \quad \text{where } a, b \text{ Const}$$

* Solution steps:

1) assume the solution is following Form:
$y = e^{mx}$, m is constant and must be determined.

2) $y = e^{mx} \rightarrow y' = m e^{mx} \rightarrow y'' = m^2 e^{mx}$

3) $m^2 e^{mx} + a m e^{mx} + b e^{mx} = 0$
$e^{mx} [m^2 + am + b] = 0$
$m^2 + am + b = 0$

معادلة من الدرجة الثانية ∴ لها جذرين هما $m_{1,2}$
∴

$$m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

m_1, m_2 has three Probabilities.

$\boxed{1}$ $m_1 \neq m_2$

∴ Solution is the form

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$\boxed{2}$ $m_1 = m_2$

∴ Solution in the form

$$y = c_1 e^{mx} + c_2 x e^{mx}$$

$\boxed{3}$ $m_1, m_2 = \alpha \pm i\beta$

الجزء الحقيقي   الجزء التخيلي

$$y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$$

Ex1: $y'' - 6y' + 5y = 0$ ← Homogeneous

Constant

1- assume the solution in the Form:

$y = e^{mx} \rightarrow \cancel{y' = e^{m}}$

$y' = m e^{mx} \rightarrow y'' = m^2 e^{mx}$

2- $m^2 e^{mx} - 6 m e^{mx} + 5 e^{mx} = 0$

$e^{mx} [m^2 - 6m + 5] = 0$

$m^2 - 6m + 5 = 0$

$(m_1 - 5)(m_2 - 1) = 0$

$m_1 = 5 \quad , \quad m_2 = 1$

$\because m_1 \neq m_2 \rightarrow$ (First Probability)

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y = c_1 e^{5x} + c_2 e^{x}$

Ex2 : $y'' - 6y' + 9y = 0$ ← Homogeneous

constant

1- assume the solution in the form

$$y = e^{mx} \rightarrow y' = m e^{mx} \rightarrow y'' = m^2 e^{mx}$$

2) $$m^2 e^{mx} - 6m e^{mx} + 9 e^{mx} = 0$$

$$e^{mx} [m^2 - 6m + 9] = 0$$

$$m^2 - 6m + 9 = 0$$

$$(m_1 - 3)(m_2 - 3) = 0$$

$$m_1 = m_2$$ (second)

$$y = c_1 e^{mx} + c_2 x e^{mx}$$

$$y = c_1 e^{3x} + c_2 x e^{3x}$$

التاريخ: / /

Ex3: $y'' + 4y' + 5y = 0$

1- assume the Solution in Form

$y = e^{mx} \rightarrow y' = me^{mx} \rightarrow y'' = m^2 e^{mx}$

2- $m^2 e^{mx} + 4me^{mx} + 5e^{mx} = 0$

$e^{mx}[m^2 + 4m + 5] = 0$

$m^2 + 4m + 5 = 0$

$$m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$m_{1,2} = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$= -2 \pm i$$

$\alpha = -2 \qquad \beta = 1$

$$y = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$$

$$y = e^{-2x}[c_1 \cos x + c_2 \sin x]$$

Ex 4 : $y'' + 25y = 0$

→ assume the solution in Form

$y = e^{mx} \rightarrow y' = m\,e^{mx} \rightarrow y'' = m^2 e^{mx}$

$\therefore m^2 e^{mx} + 25 \cdot e^{mx} = 0$

$e^{mx}[m^2 + 25] = 0$

$m^2 + 25 = 0$

$m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

$= \frac{\pm\sqrt{-100}}{2} = \pm 5i$

$m_{1,2} = \pm 5i$

$\alpha = 0 \quad , \beta = 5$

$y = e^{0x}[c_1 \cos 5x + c_2 \sin 5x]$

$y = [c_1 \cos 5x + c_2 \sin 5x$

Ex 5) $y'' + 5y' + 6y = 0$

$$y = e^{mx} \rightarrow y' = m e^{mx} \rightarrow y'' = m^2 e^{mx}$$

$$m^2 e^{mx} + 5 m e^{mx} + 6 e^{mx} = 0$$

$$e^{mx}[m^2 + 5m + 6] = 0$$

$$m^2 + 5m + 6 = 0$$

$$(m + 6)(m - 1) = 0$$

$$m_1 = -6 \quad , \quad m_2 = 1$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y = c_1 e^{-6x} + c_2 e^{x}$$

التاريخ:

Ex 6: $y'' + 3y' + 2y = 0$

$y = e^{mx} \rightarrow y' = m e^{mx} \rightarrow y'' = m^2 e^{mx}$

$m^2 e^{mx} + 3m e^{mx} + 2 e^{mx} = 0$

$e^{mx} [m^2 + 3m + 2] = 0$

$m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0$

$m_1 = -2 \quad , \quad m_2 = -1$

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y = c e^{-2x} + c_2 e^{-x}$

Ex 7: $y'' + 7y' + 12y = 0$

$$y = e^{mx} \rightarrow y' = m e^{mx} \rightarrow y'' = m^2 e^{mx}$$

$$m^2 e^{mx} + 7m e^{mx} + 12 e^{mx} = 0$$

$$e^{mx} [m^2 + 7m + 12] = 0$$

$$m^2 + 7m + 12 = 0$$

$$(m+3)(m+4) = 0$$

$$m_1 = -3 \,,\; m_2 = -4$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y = c_1 e^{-3x} + c_2 e^{-4x}$$

Ex8: $y'' + y' - 2y = 0$

$$y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$$

$$m^2 e^{mx} + m e^{mx} - 2e^{mx} = 0$$

$$e^{mx}[m^2 + m - 2] = 0$$

$$m^2 + m - 2 = 0$$

$$(m+2)(m-1) = 0$$

$$m_1 = -2, \quad m_2 = 1$$

$$y = c_1 e^{-2x} + c_2 e^{x}$$